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Nonlinear edge waves and shallow-water theory

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Nonlinear effects are considered for shallow-water edge waves on beaches with a general depth distribution. The case of uniform depth away from the shoreline is considered in detail. It is shown that the results obtained are in qualitative agreement with those obtained by Whitham (1976) using the full nonlinear theory for a beach of constant slope.

1. Introduction

In the preceding paper (Whitham 1976), both the shallow-water approximation and the full water-wave theory are used to discuss nonlinear effects in edge waves for the case of a uniformly sloping beach. In that case the shallow-water approximation gives anomalous results for the amplitude decay away from the shoreline. This is attributed to the breakdown of the approximation as the depth increases. In this note, the shallow-water theory is reconsidered for more general depth distributions which may be taken to remain finite and shallow at infinity. For finite depth, the results are similar to those of the full theory for a beach of constant slope. They differ in detail because the two cases now refer to different situations: in one the depth offshore remains small compared with the wavelength while in the other it becomes large (in which case the precise depth distribution in the deep water is irrelevant since the waves are no longer influenced by the bottom).

Even in linear theory, the shallow-water approximation has undesirable features for constant slope, since it predicts an infinite number of trapped modes at the shoreline and incoming waves with non-zero amplitude at infinity are not possible. The full linear theory predicts just a finite number of edge waves and a continuous spectrum of incoming waves (Ursell 1952). Again the differences can be resolved by taking a depth distribution which becomes constant at large distances from the shore. We discuss in some detail the spectrum of the operator associated with the linear theory and show that it has a finite number of isolated points (edge waves) and a continuous part, in agreement with the full linear theory. Then the nonlinear corrections for the lowest-order mode are developed as in the previous paper.

2. Linear theory

The shallow-water equations for a depth profile $h(y)$ are

$$\left.\begin{aligned} \phi_t+\tfrac{1}{2}\phi_x^2+\tfrac{1}{2}\phi_y^2+g\zeta &= 0,\\ \zeta_t+\{(h(y)+\zeta)\,\phi_x\}_x+\{(h(y)+\zeta)\,\phi_y\}_y &= 0,\end{aligned}\right\} \tag{1}$$

where ζ is the surface elevation, ϕ is a velocity potential for the horizontal velocity field, y denotes the offshore co-ordinate and x the longshore co-ordinate. From the linearized form of (1) we obtain the following equation for ζ:

$$(h\zeta_y)_y+h\zeta_{xx}-g^{-1}\zeta_{tt} = 0. \tag{2}$$

For a travelling-wave solution of (2) of the form $\zeta = f(y)\sin(kx-\omega t)$, f satisfies

$$Lf+\lambda f \equiv (hk^{-1}f')'+(\lambda-kh)f = 0, \tag{3}$$

where $\lambda = \omega^2/gk$. In order to describe edge waves we need to discuss the spectrum of L.

For a constant beach angle β, $h(y) = \beta y$ and we have Laguerre's equation. The spectrum is positive and discrete; the eigenvalues are $\omega^2/gk = (2n+1)\beta$. However, this leads to the various discrepancies noted above. To model more realistic depth distributions, we choose h to be an increasing function such that $h(y) \sim \beta y$ as $y\to 0$ and $h(y) = h_1$ for $y \geqslant l_1$.

The domain of L is restricted to a class of functions which are finite as $y\to 0$. The operator L is self-adjoint, therefore the spectrum is confined to the real axis.

First, there are no points of the spectrum in the range $\lambda \leqslant 0$, for in that case any solution of (3) which is regular at the origin has f and f' of the same sign; it follows, writing (3) as

$$hf' = kv, \quad v' = (kh-\lambda)f,$$

that $|f|$ increases monotonically and can not be bounded at infinity.

To find the spectrum of L in $\lambda \geqslant 0$, it is convenient to use the Liouville transformation. This is $f(y) = h^{-\frac{1}{4}}(y)\,u(s)$, where

$$s(y) = \int_0^y k^{\frac{1}{2}}h^{-\frac{1}{2}}(t)\,dt.$$

The transformed equation for $u(s)$ is

$$(L_0+\lambda)\,u \equiv u''+(\lambda-q)\,u = 0, \tag{4}$$

where
$$q = kh-\frac{1}{16}\frac{h'^2}{kh}+\frac{h''}{4k}.$$

Since the Liouville transformation is in this case unitary, the spectrum of L_0 is the same as the spectrum of L. The general qualitative behaviour of q is the following: since $h(y) \sim \beta y$ for $y\to 0$, $q(s) \sim -1/4s^2$ for $s\to 0$; $q(s)$ is an increasing function, bounded by kh_1; $q(s) = kh_1$ for $s \geqslant s_1$; $q(s)$ has just one zero, which is smaller than s_1. Thus q is a potential well of infinite depth at the origin, width $s_1 = O\{l_1k^{\frac{1}{2}}h_1^{-\frac{1}{2}}\}$ and height kh_1.

For $\lambda \geqslant kh_1$, the solutions of (4) are oscillatory at infinity, and this range gives the continuous spectrum. (The relevant theorems are in Titchmarsh 1962, §§5.6, 5.7, 5.15.) For $0 \leqslant \lambda \leqslant kh_1$, there will be point eigenvalues (edge waves), whose number increases with the 'size' of the well, which is measured by

$$s_1(kh_1)^{\frac{1}{2}} = O\{kl_1\}.$$

A natural choice for the depth distribution which incorporates the edge effect and remains shallow at infinity is

$$h(y) = \beta y \quad \text{for} \quad 0 \leqslant y \leqslant l_1, \quad h(y) = h_1 = \beta l_1 \quad \text{for} \quad y \geqslant l_1. \tag{5}$$

However, the discontinuity in h' would lead to singular functions in q, so as an example for (4) we take a smoothed version: $h(y) = \beta y$ for $0 \leqslant y \leqslant l_0$, $h(y)$ equal to a smooth increasing function for $l_0 \leqslant y \leqslant l_1$ and $h(y) = h_1$ for $y \geqslant l_1$. If $h_0 = \beta l_0$ and is assumed fixed, we have

$$q(s) = \begin{cases} \frac{1}{4}\beta^2 s^2 - 1/4s^2 & \text{for} \quad 0 \leqslant s \leqslant 2(kh_0)^{\frac{1}{2}}/\beta, \\ \text{smooth function for} & 2(kh_0)^{\frac{1}{2}}/\beta \leqslant s \leqslant s_1(\beta), \\ kh_1 & \text{for} \quad s_1(\beta) \leqslant s < \infty. \end{cases}$$

Here the size of the well is measured by $2(kh_0)^{\frac{1}{2}}/\beta$. Comparison of the potential q in the interval $0 \leqslant s \leqslant 2(kh_0)^{\frac{1}{2}}/\beta$ with the potential $\frac{1}{4}\beta^2 s^2$ for Hermite's equation confirms that for $0 \leqslant \lambda \leqslant kh_1$ there are points in the spectrum (for sufficiently small β) and that their number increases as β decreases.

The overall conclusion is that the nature of the spectrum for finite depth at infinity is the same as in the full linear theory for uniformly sloping beaches.

Finally we return to (5) and work directly with (3) to find an explicit approximation to the linear dispersion relation for the lowest edge wave. We need the solution of

$$yf'' + f' + (\omega^2/g\beta - k^2 y)f = 0, \quad 0 \leqslant y \leqslant l_1, \tag{6}$$

$$f'' + (\omega^2/gh_1 - k^2)f = 0, \quad l_1 \leqslant y < \infty. \tag{7}$$

The interesting approximation for this discussion is for small β; this corresponds to large l_1 if h_1 is kept fixed. For large l_1 the solution of (6) is assumed to be close to e^{-ky}, and $\omega^2/g\beta$ is close to k. (These are the results for $l_1 = \infty$.) So we take

$$f(y) = e^{-ky} - \epsilon \tilde{f}(y), \quad \omega^2/g\beta = k(1-\epsilon), \tag{8}$$

where ϵ will be related to l_1 in the course of the argument. Then to first order in ϵ

$$y\tilde{f}'' + \tilde{f}' + (k - k^2 y)\tilde{f} = -ke^{-ky}.$$

The solution bounded at $y = 0$ is

$$\tilde{f}(y) = -e^{-ky} \int_0^y \frac{e^{2k\eta} - 1}{2\eta}\, d\eta. \tag{9}$$

The appropriate solution of (7) is

$$f(y) = Be^{-\mu ky}, \quad \mu = (1 - \omega^2/gh_1 k^2)^{\frac{1}{2}}. \tag{10}$$

Since f and f' are continuous at $y = l_1$, the impedance $f'(l_1)/f(l_1)$ must be the same for (8) and (10). From (8) and (9),

$$f(l_1) \sim e^{-kl_1} + \epsilon e^{kl_1}/4kl_1, \quad f'(l_1) \sim -k e^{-kl_1} + \epsilon e^{kl_1}/4l_1,$$
$$f'(l_1)/f(l_1) \sim -k(1 - \epsilon e^{2kl_1}/2kl_1), \quad kl_1 \gg 1.$$

(The second terms remain small since $\epsilon e^{2kl_1}/kl_1$ is ultimately found to be small.) From (10),

$$f'(l_1)/f(l_1) = -k\mu \sim -k[1-(2kl_1)^{-1}],$$

since $\omega^2 \sim gk\beta = gkh_1/l_1$ is a sufficient approximation in μ. Therefore, for the two values of $f'(l_1)/f(l_1)$ to agree

$$\epsilon = e^{-2kl_1}, \quad \omega^2 = g\beta k(1-e^{-2kl_1}) + O(e^{-4kl_1}). \tag{11}$$

3. Nonlinear corrections

We now find the nonlinear corrections to the lowest edge-wave mode. Following Whitham (1976), we consider Stokes's expansions for ϕ and ζ in the form for a travelling wave, and take

$$\phi = a\phi^{(1)}(y,\theta) + a^2\phi^{(2)}(y,\theta) + a^3\phi^{(3)}(y,\theta) + \dots, \tag{12}$$
$$\zeta = a\zeta^{(1)}(y,\theta) + a^2\zeta^{(2)}(y,\theta) + a^3\zeta^{(3)}(y,\theta) + \dots, \tag{13}$$
$$\omega = \omega_0 + a^2\omega_2 + \dots, \tag{14}$$

where $\theta = kx - \omega t$. These are substituted in (1) to obtain equations for the successive orders.

The first-order problem is

$$(h\zeta_y^{(1)})_y - (\omega_0^2/g - k^2h)\,\zeta_{\theta\theta}^{(1)} = 0, \quad \zeta^{(1)}(0,\theta) \quad \text{finite}. \tag{15}$$

Let $\zeta^{(1)} = f^{(1)}\cos\theta$. Then $f^{(1)}$ satisfies

$$(hf_y^{(1)})_y + (\omega_0^2/g - k^2h)f^{(1)} = 0, \quad f^{(1)}(0) \quad \text{finite}. \tag{16}$$

Choose ω_0^2/g to be the lowest eigenvalue of (16), and let $E(y)$ denote the corresponding edge-wave solution for $f^{(1)}$. Notice that $E(y) \propto e^{-\mu ky}$ for $y \geq l_1$, where

$$\mu = (1-\omega_0^2/gk^2h_1)^{\frac{1}{2}}.$$

The lowest-order solution is

$$\zeta^{(1)} = E(y)\cos\theta, \quad \phi^{(1)} = -g\omega_0^{-1}E(y)\sin\theta.$$

The second-order problem takes the form

$$\left.\begin{aligned} -\omega_0\phi_\theta^{(2)} + g\zeta^{(2)} &= g^2k^2\omega_0^{-1}(m^{(2)}(y) + S^{(2)}(y)\cos 2\theta), \\ -\omega_0\zeta_\theta^{(2)} + (h\phi_y^{(2)})\,y + k^2h\phi_{\theta\theta}^{(2)} &= g\omega_0^{-1}k^2T^{(2)}(y)\sin 2\theta. \end{aligned}\right\} \tag{17}$$

Let $\zeta^{(2)} = gk^2\omega_0 m^{(2)}(y) + f^{(2)}\cos 2\theta$. Then $f^{(2)}$ satisfies

$$(hf_y^{(2)})_y - (4\omega_0^2/g - 4k^2h)f^{(2)} = k^2R^{(2)}(y), \quad f^{(2)}(0) \quad \text{finite}, \tag{18}$$

where $R^{(2)}$ is a quadratic in E and $R^{(2)} = O(e^{-2\mu y})$ as $y \to \infty$. We assume that the eigenvalues of the operator $(hf')' - (nk)^2hf$ are not integer multiples of the lowest eigenvalue ω_0^2/g. Therefore, there is a solution of the second-order problem (18) of the form

$$\zeta^{(2)} = f^{(2)}(y)\cos 2\theta + gk^2\omega_0^{-1}m^{(2)}(y), \quad \phi^{(2)} = gk\omega_0^{-1}l^{(2)}(y)\sin 2\theta,$$

where $f^{(2)}$, $m^{(2)}$ and $l^{(2)}$ are $O(e^{-2\mu ky})$ as $y\to\infty$.

The third-order problem is

$$(h\zeta_y^{(3)})_y - (\omega_0^2/g - k^2h)\,\zeta_{\theta\theta}^{(3)} = \omega_0 g^{-1}(-2\omega_2 E(y)\cos\theta + \omega_0 k^2 R^{(3)}(y)\cos\theta$$
$$+\,\omega_0 k^2 S^{(3)}(y)\cos 3\theta), \quad \zeta^{(3)}(0,\theta)\quad \text{finite.} \tag{19}$$

The forcing term in (19) proportional to $\cos 3\theta$ does not resonate, hence it gives a contribution $O(e^{-3\mu ky})$ as $y\to\infty$. The crucial part of the discussion of the nonlinear problem concerns the resonant terms in (19) proportional to $\cos\theta$. Let $\zeta^{(3)} = f^{(3)}\cos\theta$. Then $f^{(3)}$ satisfies

$$(hf_y^{(3)})_y + (\omega_0^2/g - k^2h)f^{(3)} = \omega_0 g_0^{-1}(-2\omega_2 E(y) + \omega_0 k^2 R^{(3)}(y)), \quad f^{(3)}(0)\quad \text{finite.} \tag{20}$$

In order to have a square-integrable solution which satisfies the boundary condition at the origin, the right-hand side of (20) must be orthogonal to the function $E(y)$ (see the discussion of equation (20) in the preceding paper). This orthogonality condition determines ω_2:

$$\omega_2 = \tfrac{1}{2}\gamma\omega_0 k^2, \quad \text{where} \quad \gamma = \int_0^\infty R^{(3)}(y)\,E(y)\,dy\bigg/\int_0^\infty E^2(y)\,dy. \tag{21}$$

The expression for $R^{(3)}$ in terms of E is complicated for a general $h(y)$, and in any case E is not known explicitly. But $E = O(e^{-\mu ky})$ and $R^{(3)} = O(e^{-3\mu ky})$ as $y\to\infty$, so γ will differ little from the value $\gamma = \frac{1}{2}$ obtained for the case $h(y) = \beta y$. More precisely, if $h = \beta y$ for $0 \leqslant y < l_1$, then, as shown in (11), the correction is $O(e^{-2kl_1})$.

However, finding small changes in the dispersion relation was not the object of this investigation. The questions concerned the interpretation of the behaviour of $f^{(3)}(y)$ as $y\to\infty$ and the uniform validity of the expansions.

To study the behaviour of $f^{(3)}$ as $y\to\infty$, we solve (20) by variation of parameters. The solution is

$$f^{(3)}(y) = \omega_0^2 g^{-1}k^2 E(y)\,W(y),$$

where
$$W(y) = \int_0^y \frac{1}{h(\eta)\,E^2(\eta)}\int_\eta^\infty \{\gamma E^2(\xi) - R^{(3)}(\xi)\,E(\xi)\}\,d\xi\,d\eta. \tag{22}$$

In all cases $W(y)\to\infty$ as $y\to\infty$, so the third-order terms in (12) and (13) become large compared with the first-order terms, which are proportional to $E(y)$, and the expansions are not uniformly valid as $y\to\infty$. We have

$$\zeta = a(E(y) + \omega_0^2 g^{-1}k^2a^2E(y)\,W(y))\cos\theta + \dots.$$

For large y, $E(y) \propto e^{-\mu ky}$, so this becomes

$$\zeta \sim a(e^{-\mu ky} + \omega_0^2 g^{-1}a^2k^2 e^{-\mu ky}W(y))\cos\theta + \dots.$$

The method of strained co-ordinates suggests that this is the Taylor expansion of

$$\zeta \sim a\exp\{-\mu ky + \omega_0^2 g^{-1}a^2k^2 W(y)\}\cos\theta, \tag{23}$$

and that this modified form is the correct, uniformly valid one. For the beach of constant slope discussed by Whitham, $h(y) = \beta y$, $E = e^{-ky}$, $R^{(3)} = e^{-3ky}$

and $$W(y) \sim (4k\beta)^{-1} \log ky \quad \text{as} \quad y \to \infty.$$

The logarithmic behaviour seemed unnatural and was attributed to the inadequacy of the shallow-water theory for this case. This view was confirmed, since the full water-wave theory gave $W(y) \propto y$ and could be interpreted satisfactorily as yielding an amplitude dependence in the rate of decay. We are now in a position to discuss the behaviour for more general distributions $h(y)$ which do not violate the shallow-water approximations.

The asymptotic behaviour of W is given by the first term in (22), i.e.

$$W(y) \sim \int_{\text{const}}^{y} \frac{1}{hE^2(\eta)} \int_{\eta}^{\infty} \gamma E^2(\xi)\, d\xi\, d\eta. \tag{24}$$

When $h \to h_1$ as $y \to \infty$, we have $E(y) \propto e^{-\mu ky}$, $R^{(3)}(y) \propto e^{-3\mu ky}$

and $$W(y) \sim (\gamma/2\mu k h_1)\, y.$$

This is the same type of behaviour as in the full theory and again we have a clear interpretation of the result as a nonlinear modification to the rate of the exponential decay. According to (23) the appropriate rate of decay is now

$$k\left(1 - \frac{\omega_0^2}{gk^2 h_1}\right)^{\frac{1}{2}} + \frac{\omega_0^2}{gk} \frac{\gamma}{2h_1} \left(1 - \frac{\omega_0^2}{gk^2 h_1}\right)^{-\frac{1}{2}} a^2 k^2.$$

It is interesting that the term (24) originates from the frequency correction ω_2, introduced in (14) to eliminate secular terms in the Stokes expansion, but then leads to non-uniformities in y! It is unusual in nonlinear vibration problems that terms needed to construct a uniform expansion in one variable produce non-uniformities in other variables. However, in simpler examples the region concerned is finite in space, and then all non-uniformities appear in the time variable. When Stokes expansions are used to discuss periodic solutions which represent trapped modes in infinite regions, we may expect the behaviour found here.

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